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Research Article

A Double Inequality for Gamma Function

Xiaoming Zhang[1] and Yuming Chu[2]

[1] *Haining Radio and TV University, Haining 314400, Zhejiang, China*
[2] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, Zhejiang, China*

Correspondence should be addressed to Yuming Chu, chuyuming2005@yahoo.com.cn

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Using the Alzer integral inequality and the elementary properties of the gamma function, a double inequality for gamma function is established, which is an improvement of Merkle's inequality.

1. Introduction

For real and positive values of x, the Euler gamma function Γ and its logarithmic derivative ψ, the so-called psi function, are defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

respectively. For extensions of these functions to complex variables and for basic properties, see [1].

Recently, the gamma function has been the subject of intensive research, many remarkable inequalities for Γ can be found in literature [2–21]. In particular, the ratio $(\Gamma(s)/\Gamma(r))(s > r > 0)$ have attracted the attention of many mathematicians and physicists. Gautschi [22] first proved that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\left[(1-s)\psi(n+1)\right] \tag{1.2}$$

for $0 < s < 1$ and $n = 1, 2, 3 \ldots$.

A strengthened upper bound was given by Erber [23]:

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}. \tag{1.3}$$

In [24], Kečkić and Vasić established the following double inequality for $b > a > 0$:

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-(1/2)}}{a^{a-(1/2)}}e^{a-b}. \tag{1.4}$$

In [25], Kershaw obtained

$$\begin{gathered} \exp\left[(1-s)\psi\left(x+s^{1/2}\right)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{1}{2}(s+1)\right)\right], \\ \left(x+\frac{1}{2}s\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s} \end{gathered} \tag{1.5}$$

for $x > 0$ and $0 < s < 1$.

The generalized logarithmic mean $L_p(a,b)$ of order p of two positive numbers a and b with $a \neq b$ is defined by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1, p \neq 0, \\ \dfrac{b-a}{\log b - \log a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{1.6}$$

It is well known that $L_p(a,b)$ is strictly increasing with respect to p for fixed a and b. If we denote $A(a,b) = L_1(a,b) = (a+b)/2$, $I(a,b) = L_0(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $L(a,b) = L_{-1}(a,b) = (b-a)/(\log b - \log a)$, and $G(a,b) = L_{-2}(a,b) = \sqrt{ab}$ the arithmetic mean, identric mean, logarithmic mean, and geometric mean of a and b with $a \neq b$, respectively, then

$$\min\{a,b\} < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}. \tag{1.7}$$

In 1996, Merkle [26] established

$$A(\psi(a),\psi(b)) < \frac{\log\Gamma(b) - \log\Gamma(a)}{b-a} < \psi(A(a,b)) \tag{1.8}$$

for $a, b > 0$ with $a \neq b$.

It is the aim of this paper to present the new upper and lower bounds of inequality (1.8) in terms of I and L.

2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

Lemma 2.1 (see [27, page 2670]). *If $x > 0$, then*

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2}. \tag{2.1}$$

Lemma 2.2 (see [28]). *Let $f \in C[a,b]$ be a strictly increasing function. If $1/f^{-1}$ is strictly convex (or concave, resp.), then*

$$\frac{1}{b-a}\int_a^b f(t)dt > (or <, resp.) f(L(a,b)). \tag{2.2}$$

Here, f^{-1} is the inverse of f.

Lemma 2.3. *If $x > 0$, then*

$$0 < 2\psi'(x) + x\psi''(x) < \frac{1}{x}. \tag{2.3}$$

Proof. It is well known that $\log\Gamma(x) = -\gamma x + \sum_{k=1}^{\infty}[x/k - \log(1 + (x/k))] - \log x$, where $\gamma = 0.577\,215\ldots$ is the Euler constant. Then, we have

$$\psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(k+x)^2} \tag{2.4}$$

$$\psi''(x) = -2\sum_{k=0}^{\infty}\frac{1}{(k+x)^3}. \tag{2.5}$$

From (2.4) and (2.5), we get

$$
\begin{aligned}
2\psi'(x) + x\psi''(x) &= \sum_{k=1}^{\infty} \frac{2k}{(k+x)^3} > 0,\\
2\psi'(x) + x\psi''(x) &= \sum_{k=1}^{\infty} \frac{2k}{(k+x)^3}\\
&< \sum_{k=1}^{\infty} \frac{2k}{(k-1+x)(k+x)(k+1+x)}\\
&= \sum_{k=1}^{\infty} \left[\frac{k}{(k-1+x)(k+x)} - \frac{k}{(k+x)(k+1+x)}\right]\\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1+x)(k+x)}\\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k-1+x} - \frac{1}{k+x}\right)\\
&= \frac{1}{x}.
\end{aligned}
\tag{2.6}
$$

□

Lemma 2.4. *Suppose that $b > a > 0$ and $f : [a,b] \to R$ is a twice differentiable function. If $f'(x) > 0$ and $2f'(x) + xf''(x) >$ (or $<$, resp.) 0 for $x \in [a,b]$, then there exists the inverse function f^{-1} of f and $1/f^{-1}$ is strictly convex (or concave, resp.).*

Proof. The existence of f^{-1} can be derived from $f'(x) > 0$ directly. Next, let $y = f(x)$, then simple computation yields

$$
\begin{gathered}
f'(x)\left(f^{-1}(y)\right)' = 1,\\
f''(x)\left[\left(f^{-1}(y)\right)'\right]^2 + f'(x)\left(f^{-1}(y)\right)'' = 0,\\
\left(\frac{1}{f^{-1}(y)}\right)'' = \frac{2\left[\left(f^{-1}(y)\right)'\right]^2}{\left(f^{-1}(y)\right)^3} - \frac{\left(f^{-1}(y)\right)''}{\left(f^{-1}(y)\right)^2}.
\end{gathered}
\tag{2.7}
$$

From (2.7) and $x = f^{-1}(y)$, we get

$$
\left(\frac{1}{f^{-1}(y)}\right)'' = \frac{2f'(x) + xf''(x)}{x^3\left(f'(x)\right)^3}.
\tag{2.8}
$$

Therefore, the strict convexity (or concavity, resp.) of $1/f^{-1}$ follows from (2.8) and the assumed condition $2f'(x) + xf''(x) >$ (or $<$, resp.) 0. □

3. Main Result

Theorem 3.1. *For all $a, b > 0$ with $a \neq b$, one has*

$$\psi(L(a,b)) < \frac{\log\Gamma(b) - \log\Gamma(a)}{b-a} < \psi(L(a,b)) + \log\frac{I(a,b)}{L(a,b)}. \tag{3.1}$$

Proof. Without loss of generality, we assume that $b > a > 0$. From (2.4) and Lemma 2.3, together with Lemma 2.4, we clearly see that ψ is strictly increasing and $1/\psi^{-1}$ is strictly convex on $[a, b]$. Then, Lemma 2.2 leads to

$$\frac{1}{b-a}\int_a^b \psi(t)dt > \psi(L(a,b)). \tag{3.2}$$

Therefore, the left-side inequality in (3.1) follows from (3.2).

Next, for $x \in [a, b]$, let $g(x) = \psi(x) - \log x$. Then, Lemmas 2.1 and 2.3 lead to

$$g'(x) = \psi'(x) - \frac{1}{x} > \frac{1}{2x^2} > 0, \tag{3.3}$$

$$2g'(x) + xg''(x) = 2\psi'(x) + x\psi''(x) - \frac{1}{x} < 0. \tag{3.4}$$

From (3.3) and (3.4), together with Lemma 2.4, we clearly see that $g(x)$ is strictly increasing and $1/g^{-1}$ is strictly concave on $[a, b]$. Then, Lemma 2.2 implies

$$\frac{1}{b-a}\int_a^b \bigl(\psi(t) - \log t\bigr)dt < \psi(L(a,b)) - \log L(a,b). \tag{3.5}$$

Therefore, the right-side inequality in (3.1) follows from (3.5).

To compare the bounds in Theorem 3.1 with that in (1.8), we have the following two remarks. □

Remark 3.2. The lower bound in Theorem 3.1 is greater than that in (1.8), that is, $\psi(L(a,b)) > A(\psi(a), \psi(b))$ for $a, b > 0$ with $a \neq b$. In fact, for any $b > a > 0$ and $x \in [a, b]$, Lemmas 2.1 and 2.3 lead to

$$\psi'(x) + x\psi''(x) < -\frac{1}{2x^2} < 0. \tag{3.6}$$

From (3.6) and [29], we know that $\psi(x)$ is a strictly geometric-arithmetic concave function on $[a, b]$, hence, we get

$$\psi(G(a,b)) > A\bigl(\psi(a), \psi(b)\bigr). \tag{3.7}$$

Since ψ is strictly increasing and $G(a, b) < L(a, b)$, so we have

$$\psi(L(a,b)) > \psi(G(a,b)). \tag{3.8}$$

Inequalities (3.7) and (3.8) show that $\psi(L(a,b)) > A(\psi(a), \psi(b))$ for $a, b > 0$ with $a \neq b$.

Remark 3.3. The upper bound in Theorem 3.1 is less than that in (1.8), that is, $\psi(L(a,b)) + \log I(a,b) - \log L(a,b) < \psi(A(a,b))$. In fact, for any $b > a > 0$ and $x \in [a,b]$, (3.3) and $L(a,b) < I(a,b)$ imply

$$\psi(L(a,b)) - \log L(a,b) < \psi(I(a,b)) - \log I(a,b). \tag{3.9}$$

On the other hand, the monotonicity of ψ and $I(a,b) < A(a,b)$ leads to

$$\psi(I(a,b)) < \psi(A(a,b)). \tag{3.10}$$

From (3.9) and (3.10), we get

$$\psi(L(a,b)) + \log I(a,b) - \log L(a,b) < \psi(A(a,b)). \tag{3.11}$$

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